Assignment 9

Hand in no. 1, 4, 7, and 8 by November 16.

1. Consider Theorem 3.4. Show that the function Φ maps $\overline{B_r(x_0)}$ to $\overline{B_R(y_0)}$ has an inverse, that is, there is a continuous function G in $B_R(y_0)$ back to $B_r(x_0)$ satisfying $\Phi(G(y)) = y$ for all $y \in \overline{B_R(y_0)}$.

2. Consider the function
$$h(x,y) = (x-y^2)(x-3y^2), \quad (x,y) \in \mathbb{R}^2.$$
Show that the set $\{(x,y) : \ h(x,y) = 0\}$ cannot be expressed as a local graph of a C^1-function over the x or y-axis near the origin. Explain why the Implicit Function Theorem is not applicable.

3. Consider the mapping from $\mathbb{R}^2$ to itself given by $f(x,y) = x - x^2, \ g(x,y) = y + xy$. Show that it has a local inverse at $(0,0)$. And then write down the inverse map so that its domain can be described explicitly.

   In the following the Initial Value Problem (IVP) refers to $x' = f(t,x), \ x(t_0) = x_0$, where f satisfies the Lipschitz condition in some rectangle containing (t_0, x_0) in its interior, see Notes for details.

4. Solve the (IVP) for $f(t,x) = \alpha t(1+x^2), \alpha > 0, \ t_0 = 0$, and discuss how the (largest) interval of existence changes as α and x_0 vary.

5. Optional. Deduce Picard-Lindelöf Theorem based on the ideas of perturbation of identity. Hint: Take a particular
$$y = \int_{t_0}^{t} f(t, x_0)dt$$
in the integral equation as $x + \Psi(x) = y$ for some Ψ.

6. Show that the solution to IVP belongs to C^{k+1} (as long as it exists) provided $f \in C^k$ for $k \geq 1$. In particular, $y \in C^\infty$ provided $f \in C^\infty$.

7. Let $f \in C(D), D = (a,b) \times (c,d)$, satisfying the Lipschitz condition. Let x_1 and x_2 be two solutions to (IVP) defined on open subintervals I and J in (a,b) respectively with $x_1(t_0) = x_2(t_0)$ at some $t_0 \in I \cap J$. Assuming that their graphs lie in D. Show that they are equal on $I \cap J$.

8. Fill out the details for the Picard-Lindeöf Theorem for systems (Theorem 3.12) in Notes.

9. Show that there exists a unique solution h to the integral equation
$$h(x) = 1 + \frac{1}{\pi}\int_{-1}^{1} \frac{1}{1+(x-y)^2} h(y)dy,$$
in $C[-1,1]$. Also show that h is non-negative.